

Quark number susceptibilities from resummed perturbation theory

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We evaluate the second and fourth order quark number susceptibilities in hot QCD using two variations of resummed perturbation theory. On one hand, we carry out a one-loop calculation within hard-thermal-loop perturbation theory, and on the other hand perform a resummation of the four-loop finite density equation of state derived using a dimensionally reduced effective theory. Our results are subsequently compared with recent high precision lattice data, with which they are seen to agree down to temperatures very close to the transition region.

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Introduction. One of the most important challenges in the equilibrium thermodynamics of QCD is to develop nonperturbative tools to access the region of nonzero quark densities, addressing questions such as the existence and location of a critical point on the phase diagram. Barring a solution to the sign problem of lattice QCD, the leading method to determine the finite density equation of state (EoS), or the behavior of the pressure as a function of quark chemical potentials μ , is through the evaluation of quark number susceptibilities,

$$\chi_{ijk\dots}(T) \equiv \left. \frac{\partial^n p(T, \{\mu_f\})}{\partial \mu_i \partial \mu_j \partial \mu_k \dots} \right|_{\mu_f=0}, \quad (1)$$

where the indices i, j, k, \dots refer to different quark flavors. These functions carry information about the response of the system to nonzero density, yet can be determined on the lattice without problems; for examples of recent studies, see e.g. [1, 2] and references therein. The applicability of these results to the determination of the EoS at $\mu \neq 0$ is ultimately restricted only by the convergence of the expansion of the pressure in powers of μ/T .

While a quantitative description of the quark gluon plasma near its transition temperature T_c clearly necessitates the use of nonperturbative techniques, it is also very interesting to study, to what extent the behavior of the quark number susceptibilities can be understood using analytic weak coupling methods. First, unlike lattice calculations, perturbation theory works best at very high temperatures and thus offers a way to connect the results obtained around T_c to arbitrarily high energies. More importantly, perturbative calculations are easily generalizable to finite density, and are not constrained to the region of small μ/T . Finally, due to the disappearance of *purely* gluonic contributions to the quark number susceptibilities, these quantities are expected to display improved convergence properties in comparison with the pressure itself. Indeed, extensive work on susceptibilities, and more generally the chemical potential dependence of the pressure, has been carried out within unresummed perturbation theory [3, 4], the hard-thermal-loop approximation [5–9], the analytically tractable large- N_f limit of QCD [10, 11], and even the gauge/gravity duality [12]. In addition to this, the applicability of dimensional reduction (DR) to finite densities has been investigated

in [13], and the behavior of the susceptibilities determined through a nonperturbative DR study in [14].

While many of the perturbative calculations listed above showed reasonably good agreement with lattice results existing at the time of their publication, the numerically significant corrections present in recent high precision lattice data [1, 2] clearly call for a re-examination of the issue. On top of this, the past years have witnessed important progress in the resummation of high-temperature perturbation theory on multiple fronts. In hard-thermal-loop perturbation theory (HTLpt) [15], a recent evaluation of the partition function of hot QCD up to three-loop order has demonstrated dramatically improved convergence properties [16, 17], and the agreement between the HTLpt and lattice results has subsequently been observed to be very good down to $2 - 3 T_c$ (for the relevant lattice data, see e.g. [18–20]). In addition, the same framework has been applied to the case of finite density and zero temperature, albeit at lower orders [21, 22]. At the same time, it was shown in [23, 24] that a simple resummation of the soft, three-dimensional contributions to the four-loop EoS of hot QCD [25] is enough to considerably decrease its renormalization scale dependence, resulting in excellent agreement with lattice data. It should be very interesting to see, what kind of an effect these new techniques have when applied to the evaluation of quark number susceptibilities.

In the present paper, our objective is simple: We want to apply state-of-the-art resummation techniques to the determination of the second and fourth order quark number susceptibilities, and compare the results to the most recent lattice data available. To this end, we address two separate calculations: First, we employ HTLpt to determine the susceptibilities at one-loop order, and then apply the resummation scheme of [23, 24] to the four-loop $\mu \neq 0$ EoS derived in [3] to obtain DR results for the same quantities. All of our calculations are performed within the $\overline{\text{MS}}$ scheme of dimensional regularization, denoting the renormalization scale by $\bar{\Lambda}$. Furthermore, we will in this paper specialize to the phenomenologically relevant case of three dynamical quark flavors, which for simplicity are all taken to be massless. We have, however, explicitly verified that keeping the leading order strange

quark mass dependence in the results only affects them in any noticeable way at the very lowest temperatures.

HTL perturbation theory. Hard-thermal-loop perturbation theory is a reorganization of perturbative expansions within thermal QCD. The Lagrangian density of the theory is written in the form

$$\mathcal{L}_{\text{HTLpt}} = (\mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{HTL}}) \Big|_{g \rightarrow \sqrt{\delta}g} + \Delta\mathcal{L}_{\text{HTL}}, \quad (2)$$

where \mathcal{L}_{QCD} is the usual Lagrangian density of the theory, \mathcal{L}_{HTL} the HTL improvement term, and δ a formal expansion parameter introduced for bookkeeping purposes. The last part of the Lagrangian, $\Delta\mathcal{L}_{\text{HTL}}$, on the other hand contains counterterms, which are necessary to cancel the ultraviolet divergences introduced by the HTLpt reorganization. For full QCD with dynamical quarks, the (gauge invariant) HTL improvement term reads

$$\begin{aligned} \mathcal{L}_{\text{HTL}} = & -\frac{1}{2}(1-\delta)m_D^2 \text{Tr} \left(F_{\mu\alpha} \left\langle \frac{y^\alpha y^\beta}{(y \cdot D)^2} \right\rangle_y F_\beta^\mu \right) \\ & + (1-\delta)i \sum_f^{N_f} m_{q,f}^2 \bar{\psi}_f \gamma^\mu \left\langle \frac{y_\mu}{y \cdot D} \right\rangle_y \psi_f, \end{aligned} \quad (3)$$

where $D^\mu = \partial^\mu - igA^\mu$ denotes a covariant derivative (in the appropriate representation), $y = (1, \hat{\mathbf{y}})$ is a light-like four vector, $\langle \rangle_y$ represents an average over the direction of $\hat{\mathbf{y}}$, and m_D and $m_{q,f}$ are the Debye mass and fermion thermal mass parameters. Note that $m_{q,f}$ carries dependence on the flavor index f , running from 1 to $N_f = 3$.

HTLpt is formally defined as an expansion of physical quantities in powers of δ around $\delta = 0$, implying that already at its leading order δ^0 one is dealing with dressed propagators that incorporate important plasma effects, such as Debye screening and Landau damping. The starting point of HTLpt is thus an ideal gas of massive quasiparticles, which can be identified as one of the main reasons for its success. At higher orders, the expansion in δ generates dressed vertices as well as higher order terms that ensure that there is no overcounting of Feynman diagrams.

In practice, physical observables are calculated within HTLpt by expanding the corresponding expressions in powers of δ , truncating at some specified order, and then setting $\delta = 1$. If it were possible to carry out the expansion to all orders, the final result would be independent of the parameters m_D and $m_{q,f}$. At any finite order in δ , some dependence however remains, and a prescription for choosing their values is required. Optimally, they should be determined via a variational condition for the thermodynamic potential, but at leading order, this would lead to $m_D = m_{q,f} = 0$. In our calculation, we therefore identify these parameters with their weak coupling values,

$$m_D^2 = \frac{g^2}{3} \left[\left(N_c + \frac{N_f}{2} \right) T^2 + \frac{3}{2\pi^2} \sum_f \mu_f^2 \right], \quad (4)$$

$$m_{q,f}^2 = \frac{g^2}{4} \frac{N_c^2 - 1}{4N_c} \left(T^2 + \frac{\mu_f^2}{\pi^2} \right), \quad (5)$$

where we have kept the number of colors N_c arbitrary.

After the definitions above, the one-loop HTLpt determination of the EoS follows to a large extent the $\mu = 0$ work of [15]. In the high temperature limit, we further perform an analytic expansion of our result in powers of m_D/T and m_q/T , resulting in

$$\begin{aligned} p_{\text{HTLpt}} = & \frac{d_A \pi^2 T^4}{45} \left\{ 1 + \frac{N_c}{d_A} \sum_f \left(\frac{7}{4} + 30\hat{\mu}_f^2 + 60\hat{\mu}_f^4 \right) \right. \\ & - \frac{15}{2} \hat{m}_D^2 - \frac{30N_c}{d_A} \sum_f (1 + 12\hat{\mu}_f^2) \hat{m}_{q,f}^2 \\ & + 30\hat{m}_D^3 + \frac{60N_c}{d_A} (6 - \pi^2) \sum_f \hat{m}_{q,f}^4 \\ & + \frac{45}{4} \left(\gamma_E - \frac{7}{2} + \frac{\pi^2}{3} + \log \frac{\bar{\Lambda}}{4\pi T} \right) \hat{m}_D^4 \\ & \left. + O(\hat{m}_D^6/T^6, \hat{m}_q^6/T^6) \right\}, \end{aligned} \quad (6)$$

where we have denoted $d_A \equiv N_c^2 - 1$ as well as introduced the dimensionless parameters $\hat{m} \equiv \frac{m}{2\pi T}$, etc. Results for the quark (or baryon) number susceptibilities are then obtained by taking derivatives of this expression with respect to the chemical potentials, and in the end finally setting $\mu = 0$.

Dimensional reduction. To date, the unresummed weak coupling expansion of the QCD pressure has been worked out up to and partially including its four-loop order, both at zero density [25–27] and at $\mu \neq 0$ [4]. A very useful tool in these calculations has turned out to be the three-dimensional effective theory electrostatic QCD (EQCD), the partition function of which contains the contributions of the soft and ultrasoft momentum scales (gT and g^2T , respectively) to the corresponding quantity in the full theory [28, 29]. In practice, one writes the pressure of the four-dimensional theory in the form

$$p_{\text{QCD}} = p_{\text{HARD}} + T p_{\text{EQCD}}, \quad (7)$$

where p_{HARD} is defined as the result of a *strict loop expansion* in the full theory, obtained by letting dimensional regularization regulate both the UV and IR divergences. At the same time, p_{EQCD} is obtained from the partition function of EQCD, which one can evaluate using a combination of perturbative [26, 27] and nonperturbative [30, 31] tools.

Formally, EQCD is a three-dimensional Yang-Mills theory coupled to an adjoint Higgs field A_0 , originating from the zero Matsubara mode of the four-dimensional temporal gauge field. The theory is defined by the Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{EQCD}} = & g_3^{-2} \left\{ \frac{1}{2} \text{Tr}[F_{ij}]^2 + \text{Tr}[(D_i A_0)^2] + m_E^2 \text{Tr}[A_0^2] \right. \\ & \left. + i\zeta \text{Tr}[A_0^3] + \lambda_E \text{Tr}[A_0^4] \right\} + \delta\mathcal{L}_E, \end{aligned} \quad (8)$$

where we have assumed $N_c = 3$ (for larger N_c we would have two independent quartic terms for the A_0 field), and where the last term $\delta\mathcal{L}_E$ stands for a series of higher order non-renormalizable operators that start to contribute to the EoS only beyond $\mathcal{O}(g^6)$. The theory is parametrized by four constants: The three-dimensional gauge coupling g_3 , the electric screening mass m_E , the cubic coupling $\zeta \sim \sum_f \mu_f$ (see [32] for details), as well as the quartic coupling λ_E . All of these parameters have expansions in powers of the four-dimensional gauge coupling g , and their values have been determined to the accuracy required by the four-loop evaluation of the EoS, some even beyond this (see e.g. [33]).

As discussed in [24], the above way of writing the full theory pressure suggests a very natural resummation scheme: While the unresummed weak coupling expansion is obtained by expanding the (perturbatively determined) EQCD partition function in powers of the four-dimensional gauge coupling g , one may alternatively simply skip this last step and keep p_{EQCD} a function of the effective theory parameters, writing

$$T p_{\text{EQCD}} = p_M + p_G, \quad (9)$$

where the functions p_M and p_G can be read off from eqs. (3.9) and (3.12) of [4]. In [24], this procedure was observed to lead to a considerable improvement of the convergence and renormalization scale dependence of the full theory pressure at zero chemical potential. It can, however, be applied to the case of the finite density pressure or the quark number susceptibilities with equal ease, which is what we have implemented in our calculations. An important step in this in principle straightforward exercise is to use the effective theory parameters in a form, where they have been analytically expanded in powers of μ/T ; cf. appendix D of [4] and appendix B of [34]. We refrain from writing the resulting, very long expressions here, but simply display the result of the procedure in the plots to follow.

Choice of parameters. Before proceeding to a quantitative comparison of our predictions with lattice data, we will briefly discuss our choices for the parameters appearing in the results. These include the values of the renormalization scale $\bar{\Lambda}$ and the QCD scale $\Lambda_{\overline{\text{MS}}}$, in addition to which a prescription for determining the form of the running gauge coupling must be specified. In both the HTLpt and DR calculations, we follow standard choices used in the literature, which we summarize below.

In perturbative calculations of bulk thermodynamic observables, the renormalization scale $\bar{\Lambda}$ is typically given a value of roughly $2\pi T$ and then varied by a factor of 2 in order to measure the sensitivity of the result with respect to this choice. Optimally, the central value should result from a prescription such as the Fastest Apparent Convergence (FAC) or the Principle of Minimal Sensitivity (PMS). For the HTLpt result, neither of these is however available, and hence the central value is chosen as $2\pi T$. In the DR calculation, we on the other hand follow a commonly used prescription introduced in [29]

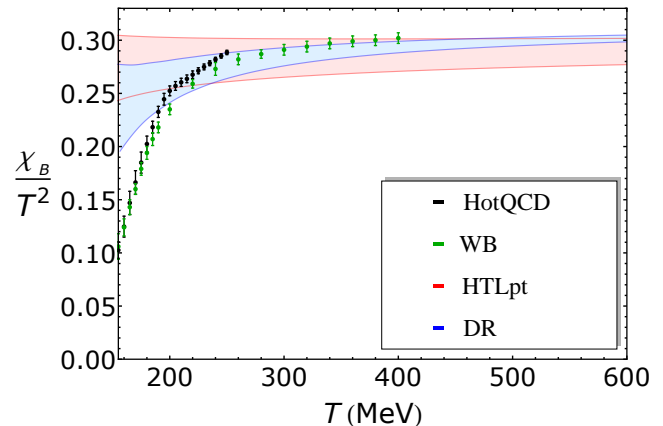


FIG. 1. A comparison of our HTLpt (wider, red band) and DR (blue band) results for the second order baryon number susceptibility χ_B/T^2 with the lattice results of the HotQCD [1] (black dots, extending to $T \approx 250$ MeV) and Wuppertal-Budapest [2] (WB, green dots) collaborations. The bands corresponding to the perturbative results originate from varying the values of $\bar{\Lambda}$ and $\Lambda_{\overline{\text{MS}}}$ within the ranges indicated in the text. Asymptotically, all of the results approach the limiting value of $1/3$.

and apply FAC to the three-dimensional gauge coupling g_3 , thus obtaining $\bar{\Lambda}_{\text{central}} \approx 1.445 \times 2\pi T$.

For the dependence of the gauge coupling constant on the renormalization scale, we use a one-loop perturbative expression in the HTLpt result and a two-loop one in the DR case. This is in accordance with the usual rule that the uncertainties originating from the running of the gauge coupling should not exceed those due to the perturbative computation itself. Finally, for the choice of the QCD scale $\Lambda_{\overline{\text{MS}}}$ we use a recent lattice determination of the value of the strong coupling constant at a reference scale of 1.5 GeV [35]. Requiring that our one- and two-loop running couplings agree with this, we obtain the values of 176 and 283 MeV in these two cases, respectively. To be conservative, we vary the value of $\Lambda_{\overline{\text{MS}}}$ around these numbers by 30 MeV, which is somewhat larger than the reported lattice error bars.

Results. In Fig. 1, we display our results for the second order baryon number susceptibility $\chi_B \equiv \partial^2 p / \partial \mu_B^2$, which to a very good accuracy satisfies the relation $\chi_B = \chi_{uu}/3$ and for which most of the lattice data has been derived. As the widths of the red and blue bands — corresponding respectively to the HTLpt and DR results — demonstrate, the dependence of our results on the renormalization scale and the value of $\Lambda_{\overline{\text{MS}}}$ is rather mild. For instance, a comparison of the DR band with the unresummed four-loop result of [3] shows a reduction of the uncertainty by a factor of nearly 10 in this temperature range. Our two results are in addition in reasonably good agreement with each other, considering that the current HTLpt result is only of one-loop order. A comparison with the recent continuum extrapolated lattice data of

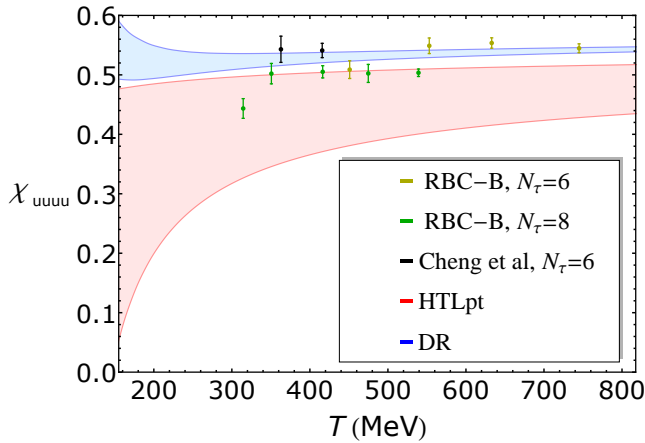


FIG. 2. A comparison of our results for the fourth order susceptibility χ_{uuuu} with the p4 lattice data of [36] (RBC-B) and [37] (Cheng et al.). In the $T \rightarrow \infty$ limit, this quantity approaches the value $6/\pi^2 \approx 0.61$.

both the HotQCD [1] and Wuppertal-Budapest [2] collaborations further shows very good agreement down to temperatures of roughly 250 MeV, or less than twice T_c .

Moving on to fourth order susceptibilities, we no longer have continuum extrapolated lattice data at our disposal. In Fig. 2, we nevertheless compare our results for the light quark number susceptibility χ_{uuuu} to the lattice data of [36, 37], obtained with the p4 action. This time, the perturbative results no longer overlap with each other, but are nonetheless both in reasonable agreement with the lattice. Our hope is that the emergence of continuum extrapolated lattice results will eventually help to clarify the situation.

Conclusions. In the present paper, we have applied

two types of resummed perturbation theory to the determination of the second and fourth order quark number susceptibilities in thermal QCD. Our main results are displayed in Figs. 1 and 2, of which in particular the former shows impressive agreement with recent lattice data down to temperatures surprisingly close to the transition region. Although the state-of-the-art perturbative determinations of the zero density EoS compare to the lattice results quite favorably, an agreement at this level is clearly exceptional (for another very successful case, see [38]). We interpret this behavior as a reflection of the fermionic nature of the susceptibilities; after all, due to the chemical potential derivatives appearing in eq. (1), the most poorly convergent, purely gluonic contributions to the EoS altogether drop out from this quantity.

The most important virtue of weak coupling methods is clearly their versatility. Indeed, as soon as the quark number susceptibilities for the three-flavor case treated here have been computed, it is next to trivial to extend the results to other theories of interest, such as two-flavor or quenched QCD (or even QCD with a different number of colors), as well as to other quantities, such as higher order susceptibilities or the pressure as a function of chemical potentials. All of these cases are examples of directions we will pursue in a forthcoming publication [39], hoping that the results will find phenomenological applications in the study of the current and future heavy ion data from RHIC, LHC and FAIR.

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